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Quasicoordinate Equations for Flexible Spacecraft

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Introduction

THE purpose of this Note is to quash the quasicoordinate controversy. For those with the responsibility for developing simulation capabilities for spacecraft of increasing flexibility and dynamic complexity, with increasingly stringent pointing requirements, and with the increasing importance of cost effectiveness calling for dramatic improvements in computational efficiency, the stories that circulate in technical discussions over the promise of the quasicoordinate approach are seductive indeed, and worthy of investigation. This Note describes the results of one such investigation, which culminated in the following proposition.

Proposition

Applying Euler's rotational equation in the vector form $\mathbf{M} = \dot{\mathbf{H}}$ *to any material continuum, where* $\mathbf{M}$ *is the moment of external forces about the system mass center and* $\dot{\mathbf{H}}$ *is the inertial time derivative of the system angular momentum about the system mass center, and recording scalar equations for an orthogonal vector basis fixed in any reference frame* f *in which the system mass center is fixed, including the inertial angular velocity* $\boldsymbol{\omega}$ *of* f *among the variables, produces the same three equations of motion that emerge from Lagrange's quasicoordinate formulation, with the scalar components of* $\boldsymbol{\omega}$ *for a vector basis fixed in* f *chosen as the quasicoordinate derivatives.*

The validity of this proposition in application to a single rigid body is well known,[1,2] but its general applicability does not appear to have been demonstrated. The importance of this proposition lies not in what it tells us to do in order to improve our spacecraft simulation programs, but in what it tells us not to do; if we are already in possession of a system of equatons or a simulation program based on a Newton-Euler formulation, we should *not* make the investment required to obtain a new set of equations or a new computer program based on a quasicoordinate formulation, and conversely.

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Theoretical Background

Although Whittaker[1] tells us that particular cases of the quasicoordinate equations were known to Lagrange and Euler, and that the general form is due to Boltzmann (1902) and Hamel (1904), still the use of quasicoordinates is not widespread. The so-called Lagrangian quasicoordinate equations† are included in recent books,[2] and modern variants of the quasicoordinate equations have been advanced,[3] but still the concept remains on the edge of memory for most dynamicists, and genuinely familiar to very few. The only application of quasicoordinates to flexible spacecraft in the literature is the interesting work of Bodley and Park,[4] in which Lagrange's quasicoordinate equations are foregone in favor of a direct D'Alembert approach employing quasicoordinates.

Lagrange's equations for generalized coordinates $q_1, \ldots, q_v$, appear in their most general form as the matrix equations

$$(d/dt)(T_{,\dot{q}}) - T_{,q} = Q - A^T\lambda \tag{1}$$

in which T is the kinetic energy expressed in terms of the scalars $q_1, \ldots, q_v$ in the column matrix q and the scalars $\dot{q}_1, \ldots, \dot{q}_v$ in the column matrix $\dot{q}$; the comma convention is used for partial differentiation (so that $T_{,q}$ and $T_{,\dot{q}}$ are $v \times 1$ matrices); Q is the $v \times 1$ matrix of generalized forces defined by

$$Q_k \triangleq \int \dot{\mathbf{R}}_{,\dot{q}_k} \cdot d\mathbf{f}, \qquad k = 1, \ldots, v \tag{2}$$

where $\dot{\mathbf{R}}$ is the inertial velocity of a differential element subjected to force $d\mathbf{f}$; λ is an $m \times 1$ matrix of Lagrange multipliers, and A is an $m \times v$ matrix appearing in constraint equations having the *simple* or *Pfaffian* form

$$A\dot{q} + B = 0 \tag{3}$$

for some $A(q, t)$ and $B(q, t)$. Equations (1) and (3) comprise a complete set of equations, but they are restricted in that they are formulated in terms of generalized coordinates, which by definition comprise a set of scalars the full knowledge of which is sufficient to establish the complete state of the dynamical system as a function of time. Equations of motion may be less complex in form and more readily solved when expressed in terms of quantities representing linear combinations of generalized velocities, such as the scalars $u_1, \ldots, u_v$ in the matrix equation

$$u = W^T\dot{q} + w \tag{4}$$

where W and w may depend upon $q_1, \ldots, q_v$ and t. The scalars $u_1, \ldots, u_v$ may not be derivatives of generalized coordinates; they are sometimes[2] called derivatives of *quasicoordinates.* After transposition of Eq. (4), it becomes apparent that W may be expressed as

$$W = u_{,\dot{q}}{}^T \tag{5}$$

Equations of motion equivalent to Eq. (1) can be formulated in quasicoordinate form as

$$\frac{d}{dt}(\bar{T}_{,u}) + W^{-1}[(u^T - w^T)W^{-1}W_{ij,q}]\bar{T}_{,u} + (W_{,t} - w_{,q}{}^T)\bar{T}_{,u} -$$
$$W^{-1}\{(u^T - w^T)W^{-1}W_{,q_k}\bar{T}_{,u}\} - W^{-1}\bar{T}_{,q} = W^{-1}(Q - A^T\lambda) \tag{6}$$

with the following notational conventions: $\bar{T}$ is the kinetic energy expressed in terms of u and q; the expression within braces is the kth element of a $v \times 1$ column matrix; the expression within square brackets is the element of a square $(v \times v)$ matrix in the ith row and jth column. Equation (6) is somewhat more general than the form of Lagrange's quasicoordinate equations normally encountered,[1,2] although its proof is straightforward.[5] This level of generality is uninspiring for obvious reasons; unless some simplification is introduced to eliminate the necessity of inverting W either numerically at each integration step or literally in advance of integration [in addition to the inversion necessitated by time-varying coefficients of u arising from $d(\bar{T}_{,u})/dt$], Eq. (6) is going to be even uglier than Eq. (1).

The most obvious specialization of Eq. (6) results from replacing Eq. (4) by $u = \dot{q}$; then Eq. (6) reduces to Eq. (1). A more useful specialization of Eq. (6) arises when Eq. (4) is replaced by

† Also known as the Boltzmann-Hamel equations.

$$u = \begin{Bmatrix} \dot{q}^1 \\ \cdots \\ \omega \end{Bmatrix} = \begin{bmatrix} U & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & W_o^T \end{bmatrix} \begin{Bmatrix} \dot{q}^1 \\ \cdots \\ \dot{q}^o \end{Bmatrix} \tag{7}$$

where ω is a 3×1 matrix whose elements comprise the inertial angular velocity scalar components of some reference frame f in which the system mass center is fixed, for a vector basis fixed in f, and q^o is a 3×1 matrix of generalized coordinates which define the orientation of f. In what follows, attention is further limited to holonomic systems, for which Eq. (3) can be integrated and the results used to reduce the generalized coordinates to n independent variables $q_1, \ldots, q_n$, of which the last three comprise q^o.

With these restrictions, Eq. (6) provides n independent equations, of which the first $n-3$ are identical to those resulting from Eq. (1) with $\lambda = 0$; the last three equations implied by Eq. (6) then become

$$(d/dt)(\bar{T}_{,\omega}) + \tilde{\omega}\bar{T}_{,\omega} = W_o^{-1} Q^o \tag{8}$$

where Q^o is the 3×1 matrix appearing as the last partition in Q. In this transition it has been recognized that the physical interpretation given q^o implies $\bar{T}_{,q^o} = 0$, and that the undifferentiated terms involving $\bar{T}_{,u}$ in Eq. (6) reduce to

$$\tilde{\omega}\bar{T}_{,\omega} \triangleq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{Bmatrix} \bar{T}_{,\omega_1} \\ \bar{T}_{,\omega_2} \\ \bar{T}_{,\omega_3} \end{Bmatrix} \tag{9}$$

The validity of Eqs. (8) and (9) in application to a single rigid body has long been recognized,[1,2] and proof of these equations in the general case considered here is entirely parallel to this special case.[5]

The proposition stated in the previous section can now be interpreted as the statement that for any material system Eq. (8) is identical to the equations of motion in the same variables obtained from

$$\{\mathbf{f}\}\cdot\dot{\mathbf{H}} = \{\mathbf{f}\}\cdot\mathbf{M} \tag{10}$$

where $\{\mathbf{f}\}$ is a 3×1 vector array of dextral orthogonal unit vectors fixed in f, $\mathbf{M}$ is the moment of external forces about the mass center c of the system, and $\dot{\mathbf{H}}$ is the inertial time derivative of the system angular momentum for c.

Proof

The generic position vector $\mathbf{R}$ in Eq. (2) can be expressed in terms of the symbols in Fig. 1 by

$$\mathbf{R} = \mathbf{R}^c + \bar{\boldsymbol{\rho}} + \mathbf{u} = \mathbf{R}^c + \boldsymbol{\rho} \tag{11}$$

where $\mathbf{R}^c$ is the inertial position vector of the system mass center, $\bar{\boldsymbol{\rho}}$ is a vector fixed in frame f, and $\mathbf{u}$ defines the generic displacement relative to f, with $\boldsymbol{\rho} = \bar{\boldsymbol{\rho}} + \mathbf{u}$. If ω is the inertial angular velocity of f, Eq. (2) and Eq. (11) combine for those elements of Q in Q^o as

$$Q_k^o \triangleq \int \frac{\partial\dot{\mathbf{R}}}{\partial\dot{q}_k^o}\cdot d\mathbf{f} = \int \frac{\partial}{\partial\dot{q}_k^o}(\dot{\mathbf{R}}^c + \boldsymbol{\omega}\times\boldsymbol{\rho} + \mathring{\mathbf{u}})\cdot d\mathbf{f} =$$

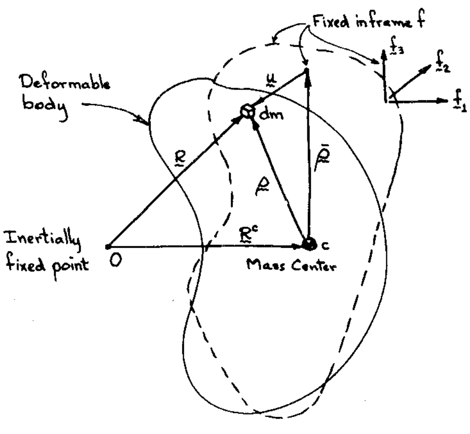

Fig. 1 Deformable body kinematics.

$$\int \frac{\partial\boldsymbol{\omega}}{\partial\dot{q}_k^o}\times\boldsymbol{\rho}\cdot d\mathbf{f} = \frac{\partial\boldsymbol{\omega}}{\partial\dot{q}_k^o}\cdot\int\boldsymbol{\rho}\times d\mathbf{f} = \frac{\partial\boldsymbol{\omega}}{\partial\dot{q}_k^o}\cdot\mathbf{M} = \omega_{,q^o}{}^T M \tag{12}$$

where open circle over a vector indicates time differentiation in frame f, and

$$M \triangleq \{\mathbf{f}\}\cdot\mathbf{M}; \qquad \omega \triangleq \{\mathbf{f}\}\cdot\boldsymbol{\omega} \tag{13}$$

Thus with Eqs. (5) and (12) the right-hand side of Eq. (8) becomes

$$W_o^{-1}Q^o = (\omega_{,\dot{q}^o}{}^T)^{-1}\omega_{,\dot{q}^o}{}^T M = M \tag{14}$$

which is by Eq. (13) identical to the right-hand side of Eq. (10). The left-hand side of Eq. (10) can be expanded with the definition

$$H \triangleq \{\mathbf{f}\}\cdot\mathbf{H} \tag{15}$$

as

$$\{\mathbf{f}\}\cdot\dot{\mathbf{H}} = \{\mathbf{f}\}\cdot(\{\mathbf{f}\}^T\dot{H} + \boldsymbol{\omega}\times\mathbf{H}) = \dot{H} + \tilde{\omega}H \tag{16}$$

Thus the left-hand sides of Eqs. (10) and (8) are identical if and only if $H = \bar{T}_{,\omega}$.

By definition

$$\mathbf{H} \triangleq \int\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}}\,dm = \int\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})\,dm + \int\boldsymbol{\rho}\times\mathring{\mathbf{u}}\,dm = \boldsymbol{\Pi}\cdot\boldsymbol{\omega} + \int\boldsymbol{\rho}\times\{\mathbf{f}\}^T\dot{u}\,dm = \{\mathbf{f}\}^T(I\omega + \int\tilde{\rho}\dot{u}\,dm) \tag{17}$$

where $\boldsymbol{\Pi}$ is the system inertia dyadic for c and $I = \{\mathbf{f}\}\cdot\boldsymbol{\Pi}\cdot\{\mathbf{f}\}^T$. Thus, from Eqs. (15) and (17),

$$H = I\omega + \int\tilde{\rho}\dot{u}\,dm \tag{18}$$

By definition

$$\begin{aligned}\bar{T} &\triangleq \tfrac{1}{2}\int\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\,dm = \tfrac{1}{2}\int(\dot{\mathbf{R}}^c + \mathring{\bar{\boldsymbol{\rho}}} + \mathring{\mathbf{u}})\cdot(\dot{\mathbf{R}}^c + \mathring{\bar{\boldsymbol{\rho}}} + \mathring{\mathbf{u}})\,dm = \\ &\tfrac{1}{2}\mathscr{M}\dot{\mathbf{R}}^c\cdot\dot{\mathbf{R}}^c + \tfrac{1}{2}\int(\boldsymbol{\omega}\times\boldsymbol{\rho})\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho})\,dm + \int(\boldsymbol{\omega}\times\boldsymbol{\rho})\cdot\mathring{\mathbf{u}}\,dm + \\ &\tfrac{1}{2}\int\mathring{\mathbf{u}}\cdot\mathring{\mathbf{u}}\,dm = \tfrac{1}{2}\mathscr{M}\dot{\mathbf{R}}^c\cdot\dot{\mathbf{R}}^c + \tfrac{1}{2}\boldsymbol{\omega}\cdot\int\boldsymbol{\rho}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})\,dm + \\ &\boldsymbol{\omega}\cdot\int\boldsymbol{\rho}\times\mathring{\mathbf{u}}\,dm + \tfrac{1}{2}\int\mathring{\mathbf{u}}\cdot\mathring{\mathbf{u}}\,dm = \tfrac{1}{2}\mathscr{M}\dot{\mathbf{R}}^c\cdot\dot{\mathbf{R}}^c + \\ &\tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\Pi}\cdot\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\{\mathbf{f}\}^T\int\tilde{\rho}\dot{u}\,dm + \tfrac{1}{2}\int\dot{u}^T\{\mathbf{f}\}\cdot\{\mathbf{f}\}^T\dot{u}\,dm\end{aligned}$$

or

$$\bar{T} = \tfrac{1}{2}\mathscr{M}\dot{\mathbf{R}}^c\cdot\dot{\mathbf{R}}^c + \tfrac{1}{2}\omega^T I\omega + \omega^T\int\tilde{\rho}\dot{u}\,dm + \tfrac{1}{2}\int\dot{u}^T\dot{u}\,dm \tag{19}$$

where $\mathscr{M}$ is the total system mass. In view of the symmetry of I, partial differentiation of $\bar{T}$ with respect to ω provides

$$\bar{T}_{,\omega} = I\omega + \int\tilde{\rho}\dot{u}\,dm \tag{20}$$

which with Eq. (18) proves that

$$H = \bar{T}_{,\omega} \tag{21}$$

and establishes the identity of Eqs. (8) and (10).

Alternative Methods

Having established that Lagrange's quasicoordinate equations in the general form of Eq. (6) produce nothing more than the combination of Lagrange's generalized coordinate equations and $\mathbf{M} = \dot{\mathbf{H}}$ in the special case of most obvious interest, as defined by Eq. (7), we might still wonder if other choices of u might be found for which Eq. (6) is superior to this alternative. The next natural candidate for u is p, the matrix of generalized momenta, as defined by

$$u = p \triangleq T_{,\dot{q}} = M\dot{q} + \Gamma \tag{22}$$

where M and Γ are defined by

$$T = \tfrac{1}{2}\dot{q}^T M\dot{q} + \dot{q}^T\Gamma + T_o$$

with M symmetric and with M, Γ, and T_o depending on q and t but not on $\dot{q}$. With the combination of Eqs. (22) and (6), we must compare the combination of Eqs. (6) and (7), which is essentially the Newton-Euler system of equations. The results should also be compared to the following:

1) The D'Alembert approach adopted by Bodley and Park,[4] which ultimately uses generalized momenta:

2) The first-order form of Eq. (1) advocated by Vance and Sitchin,[6] which may be written as the combination of Eq. (22) and

$$\dot{p} = T_{,q} + Q - A^T\lambda \tag{23}$$

3) Hamilton's equations in the form

$$\dot{p} = -\mathscr{H}_{,q} + Q - A^T\lambda \tag{24a}$$

and

$$\dot{q} = \mathscr{H}_{,p} \tag{24b}$$

where in this nonconservative context

$$\mathscr{H} \triangleq p^T\dot{q} - T$$

so that in terms of the quantities p and q, with $\bar{T} \triangleq \bar{T}(q, p, t)$,

$$\mathscr{H} = p^T\{M^{-1}(p-\Gamma)\} - \bar{T} \tag{24c}$$

and finally

4) The quasicoordinate formulation of D'Alembert's principle proposed by Kane and Wang.[7,3]

The Bodley-Park approach[4] employs initially the same quasicoordinates defined by Eq. (7), with a direct D'Alembert statement of the equations of motion. Thus at the outset their equations are implicitly the same as those considered in the previous section of this paper. They then make a transformation of variables representing the system linear and angular momenta and the generalized momenta of the modal coordinates. In terms of these new variables the equations of motion have the simple first-order form

$$\dot{y} = \bar{F}(y, \bar{\omega}, q, \dot{\bar{q}}^1) \tag{25a}$$

where the 6×1 matrix $\bar{\omega}$ of quasicoordinate derivatives contains ω as in Eq. (7) and (trivially) the system mass center linear velocity scalar components, and y is the matrix of system linear and angular momentum scalar components and the generalized momenta of deformation, and $\dot{\bar{q}}^1$ includes the system generalized coordinates associated with modal deformations. In order to complete the formulation Eq. (25a) must be complemented by the equations

$$y = \bar{M}\begin{Bmatrix} \dot{\bar{q}}^1 \\ \cdots \\ \bar{\omega} \end{Bmatrix} + \bar{\Gamma} \tag{25b}$$

where $\bar{M}$ and $\bar{\Gamma}$ are defined by

$$\bar{T} = \frac{1}{2}\begin{Bmatrix} \dot{\bar{q}}^1 \\ \cdots \\ \bar{\omega} \end{Bmatrix}^T \bar{M} \begin{Bmatrix} \dot{\bar{q}}^1 \\ \cdots \\ \bar{\omega} \end{Bmatrix} + \begin{Bmatrix} \dot{\bar{q}}^1 \\ \cdots \\ \bar{\omega} \end{Bmatrix}^T \bar{\Gamma} + \bar{T}_o$$

where $\bar{M}$ is symmetric and $\bar{M}$, $\bar{\Gamma}$ and $\bar{T}_o$ may depend on q and t but not on $\dot{\bar{q}}^1$ or $\bar{\omega}$. Equation (25b) defines the momenta in y in terms of $\bar{\omega}$ and the generalized velocities of deformation in $\dot{\bar{q}}^1$. This equation must be inverted at every integration step because $\bar{M}$ depends on $q(t)$. Equations (25a) and (25b) do not fully characterize all system generalized coordinates until they are augmented by an equation having the structure‡

$$\{\bar{\omega}\} = [\bar{W}^{o^T}]\{\dot{\bar{q}}^o\} \tag{25c}$$

where

$$\{\dot{q}\} = \begin{Bmatrix} \dot{\bar{q}}^1 \\ \cdots \\ \dot{\bar{q}}^o \end{Bmatrix}$$

It has been suggested by Vance and Sitchin that the combination of Eqs. (22) and (23) has computational advantages over Lagrange's generalized coordinate equations in the normal form represented here by Eq. (1); this combination is a first-order form of Lagrange's generalized coordinate equations, involving $T(q, \dot{q}, t)$ rather than the $\bar{T}(q, p, t)$ that appears in Eqs. (24) and Eq. (6). The combination of Eqs. (22) and (23) differs significantly from Eq. (25) primarily in the appearance of *both* generalized momenta *and* the quasicoordinates of Eq. (7) in the latter. Whereas one must invert the $n \times n$ matrix $\bar{M}$ and the 6×6 matrix $\bar{W}_o^T$ at each integration step in Eqs. (25b) and (25c) for the Bodley-Park method,§ one must invert the $n \times n$ matrix M in Eq. (22) at each step in the Vance-Sitchin approach, and one must in the Newton-Euler approach invert a highest-derivative coefficient matrix of the same dimensions, this matrix being generally simpler than M in three of its rows and columns but otherwise identical. Any of these options is however preferable to the Lagrangian quasicoordinate method of Eqs. (22) and (6) and to the Hamiltonian formulation of Eq. (24), because the presence of partial derivatives of $\bar{T}$ and $\mathscr{H}$ in these equations imposes a requirement for literal (nonnumerical) inversion of the $n \times n$ matrix M or of W^T, or alternatively the awkward numerical expansion typified by

$$M_{,q_\alpha}{}^{-1} = -(M)^{-1}M_{;q_\alpha}(M)^{-1} \tag{26}$$

Finally, it should be noted that in any application Lagrange's quasicoordinate equations must compete with the modern quasicoordinate equations proposed by Kane and Wang in 1965.[7,3] Roughly speaking, this method bears the same relationship to Lagrange's generalized coordinate form of D'Alembert's principle that Lagrange's quasicoordinate formulation bears to Lagrange's generalized coordinate equations; in application to holonomic systems with independent generalized coordinates the two sets of generalized coordinate equations are identical, as are the two quasicoordinate formulations, but for simple (Pfaffian) nonholonomic systems it is necessary to augment the state with Lagrange multipliers in the two Lagrangian methods identified here by Eqs. (1) and (6), whereas the two methods stemming directly from D'Alembert's principle do not involve Lagrange multipliers, and thus involve fewer unknowns in the state variable. For this reason, and because of an appealing simplicity in derivation procedure, Kane's modern quasicoordinate method seems generally superior to that of Lagrange, although the resulting equations are identical for holonomic systems.

‡ Here we assume that the generalized coordinates of deformation in $\bar{q}^1$ are measured with respect to a reference frame in which the mass center is fixed, so that deformations do not contribute to system linear momentum.

§ In many applications the generalized coordinates in $\bar{q}^o$ are irrelevant, and Eq. (25c) can be ignored; then $\bar{F}$ in Eq. (25a) is a function of y, $\bar{\omega}$, $\bar{q}^1$ and $\dot{\bar{q}}^1$ only.

Conclusions

With the most commonplace choice of quasicoordinates [see Eq. (7)], Lagrange's quasicoordinate equations produce precisely the same equations of motion for any material system as would emerge from the combination of Lagrange's generalized coordinate equations and the vector rotational equation $\mathbf{M} = \dot{\mathbf{H}}$. This investigator has been unable to find any other generic choice of quasicoordinates that offers any demonstrable advantage in the simulation of flexible spacecraft either, and he finds himself drawn to the conclusion that the much-discussed apparent advantages of structure of Lagrange's quasicoordinate equations disappear when the issue is examined carefully; only subjective judgments of ease of application remain for debate. This investigation invites the general conclusion that further argument over equation-formulation procedures is irrelevant, and only the choice of better mathematical models, better coordinate systems, and better programing algorithms can lead to more efficient simulations.

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